FILTER BANK DESIGN USING NILPOTENT MATRICES

Gerald Schuller *Wim Sweldens*

Lucent Technologies Bell Laboratories
600 Mountain Avenue, Murray Hill, NJ 07974
{schuller,wim}@bell-labs.com

ABSTRACT

We present a design method for filter banks with unequal length of the impulse responses for the analysis and synthesis part. This is useful e.g. for audio coding applications. A further advantage of the design method is the possibility to explicitly control the overall system delay of the filter bank, when causal filters are desired. The design method is based on a factorization of the polyphase matrices into factors with nilpotent matrices. These factors guarantee mathematical perfect reconstruction of the filter bank, and lead to FIR filters for analysis and synthesis. Using matrices with nilpotency of higher order than 2 leads to FIR filter banks with unequal filter length for analysis and synthesis.

1. INTRODUCTION

Many applications, particularly speech and audio coding, require a time frequency representation of a signal (analysis), and reconstructing the signal from this representation (synthesis). Traditionally block transforms have been used, but by now it is well known that filter banks are both more general and more powerful.

In audio coding, filter banks are used to obtain a redundancy reduction, as well as a irrelevance reduction through the application of perceptual models. The perceptual model generates the limits, below which the quantization error is inaudible. E.g. the quantization error before an acoustical event like a click or an attack has to be much smaller than after that event, in order to be inaudible. Otherwise the quantization error may be audible as a “pre-echo”. For this reason filter banks with non-symmetric impulse responses or with a low system delay are desirable. Further a possibility to obtain filter banks with unequal impulse response lengths for analysis and synthesis improves the flexibility to adapt to the perceptual limits. So can long analysis filters lead to a good frequency selectivity for a good redundancy reduction, and short synthesis filters to a limited temporal noise spread and improved irrelevance reduction.

Most existing design methods lead to orthogonal filter banks, in which case analysis and synthesis filters are time reversed versions of each other. Orthogonal filter banks suffer from three important disadvantages in audio coding: (1) Both analysis and synthesis filters have to have the same length L. (2) A delay proportional to the length of the filters ($L-1$) is needed to obtain a causal system. (3) Orthogonal filters spread the quantization noise symmetrically in time around an acoustic event. This is not a good match to the psycho-acoustic properties of the ear, cf. the pre-echo problem.

In this paper we present a general method for building biorthogonal filter banks using nilpotent matrices which avoids these problems. (1) The design method allows for different lengths in analysis and synthesis filters while both can be applied at *the cost of the shortest one*. (2) It allows careful control over the delay. (3) The resulting filter banks enable a spread of the quantization noise non-symmetrical around some signal event.

To obtain unequal filter lengths a direct approach as described by Nayebi in [1] could be used. But the presented formulation has the advantage that it is a compact mathematical description and leads to some further insights. It also provides an N-band extension of the lifting scheme, and leads to a structure for an implementation.

2. POLYPHASE DESCRIPTION

For an N-band analysis/synthesis filter bank, the input is represented by an N-dimensional vector $\mathbf{x}(m)$ composed of the downsampled input components

$$\mathbf{x}(m) = [x(mN+N-1), x(mN+N-2), \ldots, x(mN)]^t.$$

Its z-transform is the vector $\mathbf{X}(z)$. The polyphase description for an N-band filter bank with input signal $\mathbf{X}(z)$, the subband signal $\mathbf{Y}(z)$, and the reconstructed signal $\hat{\mathbf{X}}(z)$ is

$$\mathbf{Y}(z) = \mathbf{E}(z)\mathbf{X}(z) \quad \text{and} \quad \hat{\mathbf{X}}(z) = \mathbf{R}(z)\mathbf{Y}(z).$$

Here $\mathbf{E}(z)$ is the $N \times N$ analysis polyphase matrix, $\mathbf{R}(z)$ the synthesis polyphase matrix. The filter bank is perfect reconstructing (PR) if $\mathbf{R}(z) = z^{-d}\mathbf{S}^{n_t}(z)\mathbf{E}^{-1}(z)$, where $\mathbf{S}$ is a shift matrix, which shifts the elements of a signal vector by one sample:

$$\mathbf{S}(z) := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ 0 & \vdots & \vdots & \ddots & 1 \\ z & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The output then is delayed d signal blocks of length N minus n_t samples from the shift matrix, to allow for causal filters. Moreover there is an additional blocking delay of $N-1$ samples, resulting from assembling signal blocks of length N. The total system delay is thus $d \cdot N - n_t + N - 1$ samples.

Note that the elements of the polyphase matrix are Laurent polynomials, i.e. polynomials in both z and z^{-1}. For FIR filters causality can always be obtained with a suitable finite delay.

3. NILPOTENT MATRICES

In this section we mention some general properties of nilpotent matrices.

Definition 1 *We say that a square matrix* $\mathbf{A}$ *is nilpotent of order* l *(*$l > 1$*) in case* $\mathbf{A}^l = \mathbf{0}$*, where* l *is the smallest integer with this property.*

It is clear from the definition that a nilpotent matrix has determinant zero, and that all eigenvalues are zero. Traditionally, a nilpotent matrix $\mathbf{A}$ is characterized by its Jordan normal form [8, 3]:

$$\mathbf{A} = \mathbf{T}\,\mathbf{J}\,\mathbf{T}^{-1}$$

with $\mathbf{T}$ non singular,

$$\mathbf{J} = \begin{bmatrix} \mathbf{D}_1 & & \\ & \mathbf{D}_2 & \mathbf{0} \\ \mathbf{0} & & \ddots \end{bmatrix} \quad \text{and} \quad \mathbf{D}_k = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ 0 & & & \ddots \end{bmatrix}.$$

Here $\mathbf{D}_k$ is a size $n_k \times n_k$ matrix with ones on the first upper diagonal and zeros everywhere else, so that $\sum_k n_k = N$. It is easy to see that $\mathbf{D}_k$ is nilpotent of order n_k. The nilpotency order l of $\mathbf{A}$ is thus determined by the biggest matrix $\mathbf{D}_k$, $l = \max_k(n_k)$. Since $n_k \leq N$ it follows that $l \leq N$.

In this paper we particularly will use matrices of the form $\mathbf{I} + \mathbf{A}$ where $\mathbf{A}$ is nilpotent of degree l. The determinant of such a matrix is always equal to one and its inverse can immediately be found as

$$(\mathbf{I} + \mathbf{A})^{-1} = \mathbf{I} + \sum_{i=1}^{l-1} (-\mathbf{A})^i. \qquad (1)$$

Formally one can think of this as a Taylor expansion with only $l-1$ terms, as higher powers of $\mathbf{A}$ are zero.

4. FACTORIZATION

A common approach in filter bank design is to build the polyphase matrix as a product or cascade of simple, canonical matrices which are easily inverted, e.g., the lattice factorization [10] in the orthogonal case. The inverse can then easily be found by inverting each of the building blocks.

Here we consider building blocks of the type $\mathbf{I} + \mathbf{A}(z)$, where $\mathbf{A}(z)$ is nilpotent of order l. The inverse of each building block is then immediately given by (1). This guarantees that if $\mathbf{I} + \mathbf{A}(z)$ consists of Laurent polynomials (FIR filters) then so does its inverse; similarly if the building block only has causal filters again so does its inverse.

Clearly we are interested in matrices $\mathbf{A}(z)$ which are inexpensive to apply. Therefore we restrict ourselves to matrices $\mathbf{A}(z)$ of the form $z^p \cdot \mathbf{A}$ with $\mathbf{A}$ a nilpotent matrix and $p \in \{1, -1\}$. This leaves two types of matrices:

$$\mathbf{L}(z) := \mathbf{I} + z^{-1}\mathbf{A} \quad \text{and} \quad \mathbf{H}(z) := \mathbf{I} + z\mathbf{A}. \qquad (2)$$

The polyphase matrix can then be written as the products

$$\mathbf{E}(z) = \mathbf{V} \prod_{i=1}^{\nu} \mathbf{L}_i(z) \prod_{j=1}^{\mu} \mathbf{H}_j(z) \mathbf{S}^{n_a}(z). \qquad (3)$$

Each $\mathbf{L}_i(z)$ and $\mathbf{H}_j(z)$ can have a different matrix $\mathbf{A}$, and $\mathbf{V}$ is an invertible matrix. The degree of the matrices $\mathbf{L}(z)$ and $\mathbf{H}(z)$ is 1 while from (1) we see that the degree of their inverse is $l-1$. This means their use for the construction of polyphase matrices leads to FIR filter banks with *different* filter lengths for analysis and synthesis. Given that $l < N$, N has to be at least 3 for this to happen. Observe that cosine modulated filter banks can be described as a system with nested 2-band filter banks [10]. Hence they have to have the same filter length for analysis and synthesis.

The additional system delay needed to make the $\mathbf{H}_j(z)$ and $\mathbf{H}_j(z)^{-1}$ matrices causal is now equal to $l_j N$, where l_j is the nilpotency order of $\mathbf{A}_j$. Thus the total system delay of (3) including the shift matrix and blocking delay is $N - 1 - n_a - n_s + \sum_{j=1}^{\mu} l_j N$.

To better understand what happens, we consider a nilpotent matrix $\mathbf{C}$ of order 3 and let $\mathbf{E}(z) = \mathbf{I} + z^{-1}\mathbf{C}$. Then we know from (1) that $\mathbf{E}(z)^{-1} = \mathbf{I} - z^{-1}\mathbf{C} + z^{-2}\mathbf{C}^2$. Thus the analysis filters have length 2 while the synthesis filters have length 3. If it is desired that the analysis has the longer filters, the roles of $\mathbf{E}(z)$ and $\mathbf{R}(z)$ can be switched.

For implementation the matrix $\mathbf{I} + z^p\mathbf{A}$ can be brought into its Jordan form:

$$\mathbf{I} + z^p\mathbf{A} = \mathbf{I} + z^p\mathbf{T}\mathbf{J}\mathbf{T}^{-1} = \mathbf{T}(\mathbf{I} + z^p\mathbf{J})\mathbf{T}^{-1}. \qquad (4)$$

Each block $(\mathbf{I} + z^p\,\mathbf{D}_k)$ of the matrix $(\mathbf{I} + z^p\mathbf{J})$ can be factorized into the product

$$(\mathbf{I} + z^p\mathbf{D_k}) = \prod_{1 \leq i < n_k} (\mathbf{I} + z^p\mathbf{e}_{i,i+1}),$$

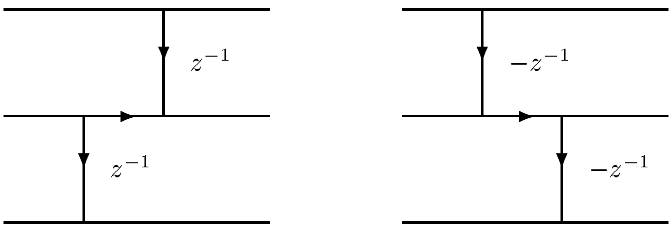

Figure 1: $(\mathbf{I}+z^{-1}\mathbf{D}_k)$ (left) and inverse (right) for $n_k = 3$.

(where $\mathbf{e}_{i,j}$ is a matrix with 1 at position (i, j) and zero everywhere else) and can thus be replaced by a structure with elementary matrices which have nilpotency of degree 2. The product can now be translated into a signal flow structure, as seen in Fig. 1 for the case $p = -1$. It is easy to see that the right structure is the inverse of the left structure. The structure also shows why in this case the right side leads to longer filters. On the left each signal path only has one delay, because of the signal flow from left to right, whereas on the right the two delays are cascaded for the signal coming from the upper left side and going to the lower right side. Observe that even though the analysis and synthesis filters have different length, there is no *difference in cost applying them*.

Using (4) one can see that the product can be built using structures as in Fig. 1 with constant invertible matrices in between, as well as in the beginning and the end.

$$\mathbf{E}(z) = \mathbf{T}_0(\prod_r(\mathbf{I}+z^{p_r}\mathbf{J})\mathbf{T}_r)\mathbf{S}^{n_a}(z) \quad \text{with} \quad p_r \in \{-1, 1\}. \tag{5}$$

Note that the factorization (3) first has all matrices of type $\mathbf{L}(z)$ and then $\mathbf{H}(z)$. In the next section we sketch a completeness proof in case one allows those factors to mix. Fig. 2 and 3 show an example for an implementation of a filter bank for $N = 5$ bands, with order of the nilpotency $l = 3$, and with $n_a = 0$ in (5). In the example the synthesis filters have longer impulse responses than the analysis filters.

5. COMPLETENESS PROOF

The proof is essentially a variant of the Smith normal form [10] for matrices $\mathbf{E}(z)$ with Laurent polynomials and with determinant one. This can always be obtained by factoring out a suitable delay z^{-d} and a shift matrix $\mathbf{S}^{n_a}(z)$. By running the Euclidean algorithm for Laurent polynomials (see, e.g. [2]) between two elements $a(z)$ and $b(z)$, they can be reduced to $c(z)$ and 0 using elementary operations where $c(z)$ is the common factor (not counting powers of z). We now run this between the first and second element of the first row (resp. column), then between the first and third, etc. Given that no row or column can have a common factor, the first row and column can be reduced to a 1 followed by all zeros. By repeating this procedure and observing that the determinant is one, the entire matrix can be reduced to the identity. The original matrix $\mathbf{E}(z)$ can thus be written as a product of matrices of the form $\mathbf{I} + a(z)\mathbf{e}_{i,j}$ with some Laurent polynomial $a(z) = \sum_k a_k z^k$. This factorization fits precisely in the lifting scheme framework considered in [9, 2]. These matrices are nilpotent, but can still have high polynomial degree.

To reduce the degree of the factors to one, each factor can be replaced by $\prod_k(\mathbf{I} + a_k z^k \mathbf{e}_{i,j})$. To show that matrices with higher powers of z can be built with matrices with lower powers it suffices to consider the 2×2 case. First note that

$$\begin{bmatrix} 1 & 0 \\ z^{-1} & 1 \end{bmatrix}\begin{bmatrix} 1 & -z \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ z^{-1} & 1 \end{bmatrix} = \begin{bmatrix} 0 & -z \\ z^{-1} & 0 \end{bmatrix}.$$

Thus the anti-diagonal matrix on the right can be build with elementary matrices of degree 1. We can now use this anti-diagonal matrix to to obtain higher powers of z:

$$\begin{bmatrix} 0 & -z \\ z^{-1} & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ z^p & 1 \end{bmatrix}\begin{bmatrix} 0 & -z \\ z^{-1} & 0 \end{bmatrix} = \begin{bmatrix} -1 & z^{p+2} \\ 0 & -1 \end{bmatrix}$$

Thus this operation increases the exponent p by 2. By switching z and z^{-1} in the anti-diagonal matrix, the exponent is decreased by 2. When the center matrix is upper triangular the exponent is also decreased by 2. By repetitive application of these principles any integer power of z can be obtained, starting with $p = 1$ or $p = 0$. If we collect all constants into matrices $\mathbf{T}_r$ we obtain (5). This concludes the sketch of the proof.

6. CONCLUSION

The use of nilpotent matrices leads to a simple design method for bi-orthogonal N-band FIR filter banks with perfect reconstruction. Our method allows for filter banks with different lengths and magnitude responses for analysis and synthesis. It is shown that the number of bands N has to be greater than or equal to 3 for this to happen. Since cosine modulated filter banks can be represented by a set of 2 band filter banks, they have to have the same filter lengths in analysis and synthesis. Further causal filter banks with a system delay independent of the filter length can be designed.

7. REFERENCES

[1] A. Akansu, M.J.T. Smith (editors) *Subband and Wavelet Transforms* Kluwer Academic Publishers, Norwell, MA, 1996.

[2] I. Daubechies and W. Sweldens. Factoring wavelet transforms into lifting steps. *J. Fourier Anal. Appl.*, 4(3):247–269, 1998.

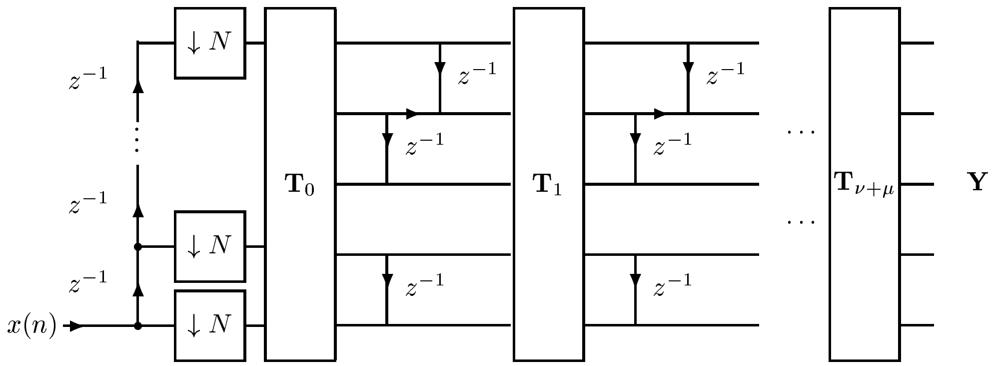

Figure 2: The analysis filter bank for $N = 5$ and nilpotency $l = 3$.

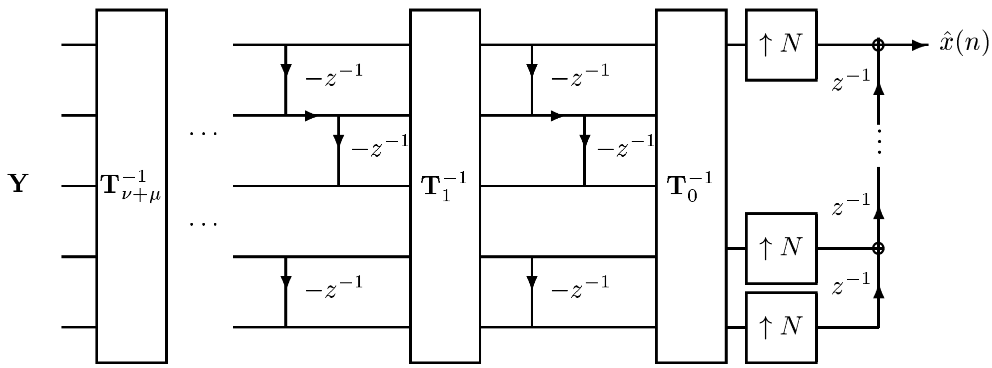

Figure 3: The synthesis filter bank.

[3] W. Greub. *Linear Algebra*. Springer Verlag, New York, 1975.

[4] T. A. C. M. Kalker and I. Shah. Ladder Structures for multidimensional linear phase perfect reconstruction filter banks and wavelets. In *Proceedings of the SPIE Conference on Visual Communications and Image Processing* (Boston), pages 12–20, 1992.

[5] S.-M. Phoong, C. W. Kim, P. P. Vaidyanathan, and R. Ansari. A new class of two-channel biorthogonal filter banks and wavelet bases. *IEEE Trans. Signal Process.*, 43(3):649–665, 1995.

[6] G. Schuller and T. Karp. Causal FIR filter banks with arbitrary system delay. In *DSP Workshop, Bryce Canyon, Utah*, 1998.

[7] G. D. T. Schuller and M. J. T. Smith. New framework for modulated perfect reconstruction filter banks. *IEEE Trans. Signal Process.*, 44(8):1941–1954, 1996.

[8] J. Stoer and R. Bulirsch. *Introduction to Numerical Analysis*. Springer Verlag, New York, 1980.

[9] W. Sweldens. The lifting scheme: A custom-design construction of biorthogonal wavelets. *Appl. Comput. Harmon. Anal.*, 3(2):186–200, 1996.

[10] P. P. Vaidyanathan. *Multirate Systems and Filter Banks*. Prentice Hall, Englewood Cliffs, NJ, 1992.